Critical fixed points in class D superconductors

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We study in detail a critical line on the phase diagram of the Cho-Fisher network model separating three different phases: metallic and two distinct localized phases with different quantized thermal Hall conductances. This system describes noninteracting quasiparticles in disordered superconductors that have neither time-reversal nor spin-rotational invariance. We find that in addition to a tricritical fixed point W_T on the critical line, separating two localized phases, there exists an additional repulsive fixed point W_N (where the vortex disorder concentration $W_N < W_T$), which splits RG flow into opposite directions: toward a clean Ising model at $W=0$ and toward W_T.

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The properties of quasiparticles in disordered superconductors belonging to new symmetry classes,[1] in particular, transitions between metallic, localized, or quantized Hall phases,[2–5] have been intensively studied. The symmetry class D may be realized in superconductors with broken time-reversal and spin-rotation invariances, as in $d+id$ superconductors with spin-orbit scattering, or in $p+ip$ superconductors, where the spin-rotational symmetry is broken by the triplet nature of the condensate. The associated changes in quasiparticle dynamics should be examined by energy transport, since neither charge density nor spin are conserved. In this Brief Report we present a detailed study of the critical line on the phase diagram for a model first introduced by Cho and Fisher (CF) (Ref. 6) (see a detailed description below), which has a particularly rich phase diagram in two dimensions.[5,7]

We employ advanced numerical calculations proposed in Ref. 5 and described in detail in Refs. 8 and 9 to overcome round-off errors in calculations of renormalized localization lengths. We also apply an optimization algorithm to determine both the critical exponent and the critical energy. These results allow us to determine the tricritical point where three phases meet and to study the dependence of the critical exponent for the insulator-to-insulator transition on the width of the system. Our results suggest the existence of two fixed points.

The original network model[10] was proposed to describe transitions between plateaux in the quantum Hall effect. In the model flux probabilities move along unidirectional links forming closed loops in analogy with semiclassical motion of electrons on contours of constant potential. Scattering between links is allowed at nodes in order to map tunneling through saddle-point potentials. Propagation along links is described by diagonal matrices with elements in the form $\exp(i\phi)$. The transfer matrix for one node relates a pair of incoming and outgoing amplitudes on the left to a corresponding pair on the right; it has the form

$$\mathbf{T} = \begin{pmatrix} \sqrt{1+\exp(-\pi\epsilon)} & \exp(-\pi\epsilon/2) \\ \exp(-\pi\epsilon/2) & \sqrt{1+\exp(-\pi\epsilon)} \end{pmatrix}, \quad (1)$$

where ϵ is a dimensionless relative distance between the electron energy and the barrier height. It is easy to see that the most "quantum" case (equal probabilities to scatter to the left and to the right) is at $\epsilon=0$.

Numerical simulations on the network model are performed on a system with fixed width M and periodic boundary conditions in the transverse direction. By multiplying transfer matrices for N slices and then diagonalizing the resulting total transfer matrix, it is possible to extract the smallest Lyapunov exponent λ [the eigenvalues of the transfer matrix are $\exp(\lambda N)$]. The localization length ξ_M is proportional to $1/\lambda$. Renormalized localization lengths for different system widths and different energies satisfy a one-parameter scaling

$$\frac{\xi_M}{M} = f\left(\frac{M}{\xi(\epsilon)}\right), \quad (2)$$

where the parameter ξ can be considered as the thermodynamic localization length. Indeed, at the critical energy ϵ_{cr}, the renormalized localization length does not depend on the system width M. This is achieved by the divergence of ξ at ϵ_{cr} on the right-hand side of Eq. (2).

For a class D symmetry a Bogoliubov–de Gennes Hamiltonian is written in terms of a Hermitian matrix.[1] The corresponding time evolution operator is real and the generalized

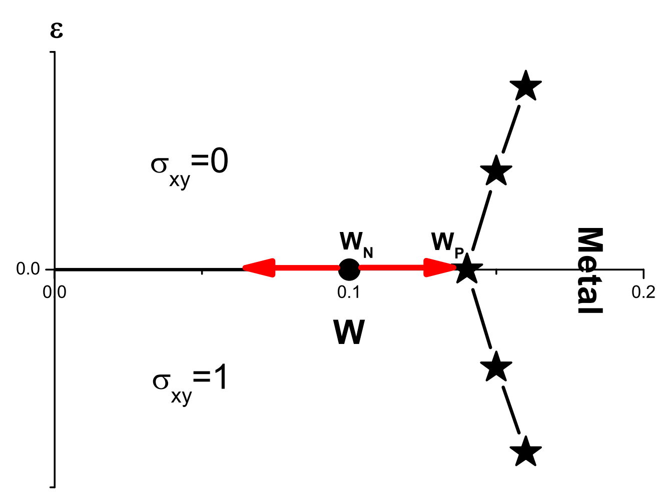

FIG. 1. (Color online) Updated phase diagram for a CF model with metallic, insulating, and quantized Hall phases.

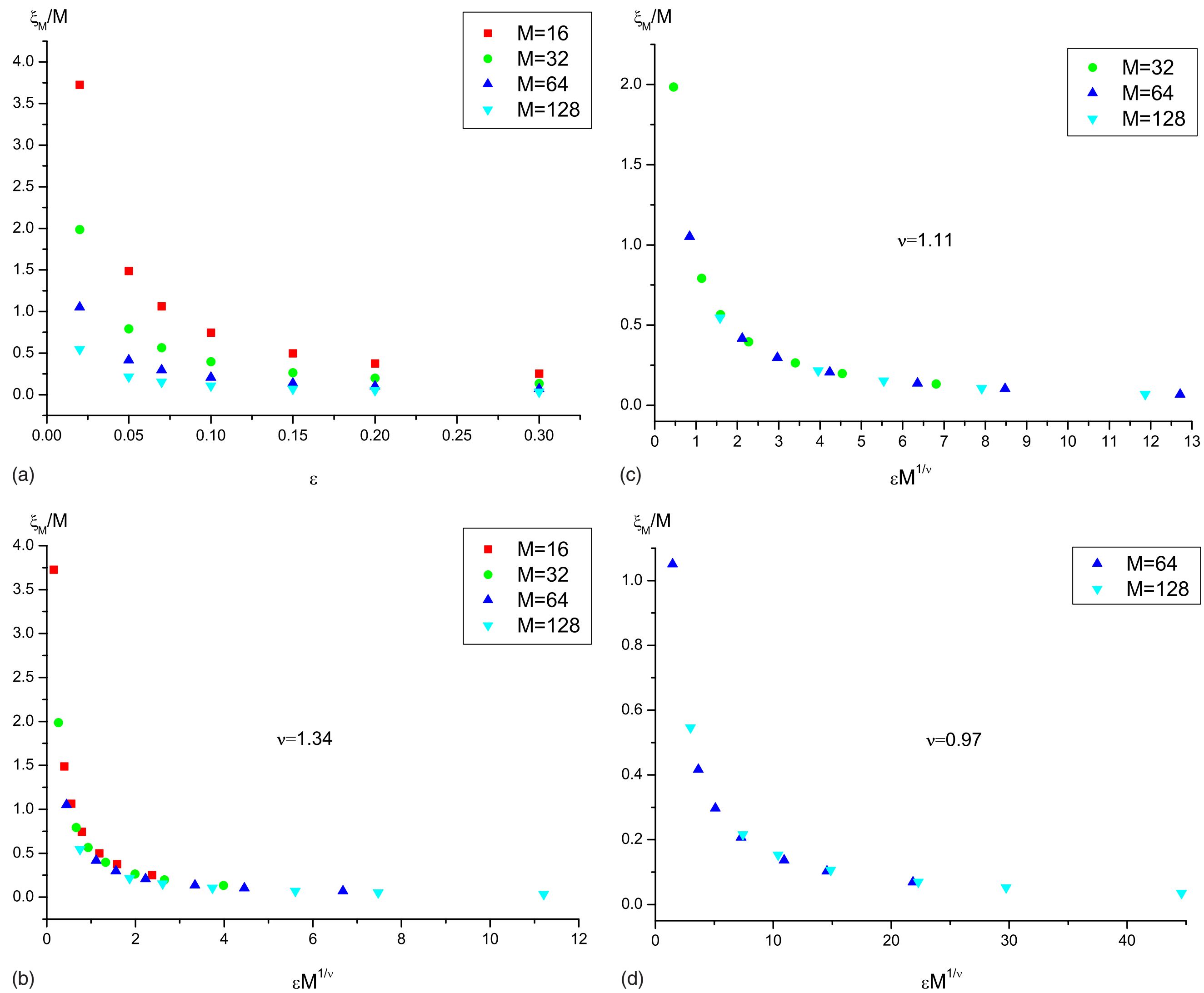

FIG. 2. (Color online) Numerical results for the disorder $W=0.04$. (a) Renormalized localization length ξ_M/M versus energy ϵ for various system widths. (b) A fit of data to one-parameter scaling form $\xi_M/M=f(\epsilon M^{1/\nu})$ for all system widths $M=16,32,64,128$ and $\nu=1.34$. (c) The same as previous for system widths $M=32,64,128$ and $\nu=1.11$. (d) The same as previous for system widths $M=64,128$ and $\nu=0.97$.

phase factors are, therefore, O(N) matrices for a model in which N-component fermions propagate on links. We study the case $N=1$ with phase factors ± 1. There are three models: random bond Ising model,[11,12] supporting two different localized phases, uncorrelated O(1) model,[13] where phases on the links are independent random variables and all states are extended,[5] and the model first introduced by CF (Ref. 6) where scattering phases with the value π appear in correlated pairs. Each model has two parameters: the first one is a disorder described by a probability W to have a phase 0 and a probability $(1-W)$ to have a phase π on a given link. The second parameter is an energy ϵ describing scattering at the nodes. For the CF model, the phase diagram (updated version of which is presented in Fig. 1) in the ϵ-W plane has three distinctive phases: metallic, and two insulating phases characterized by different thermal Hall conductances. The sensitivity to the disorder is a distinctive feature of class D.

In the CF model the disorder is introduced only at the nodes allowing for the off-diagonal elements of Eq. (1) to be multiplied by ± 1 (disorder probability W is the probability of that factor to be -1). In our previous work[14] we have studied a CF phase diagram and its critical exponents far from $\epsilon=0$. On the other hand, the critical line itself at $\epsilon=0$ is of particular interest to us. One of the possibilities discussed in Ref. 15, where a quasiparticle density of states for the CF model was studied, is the existence of the second fixed point. This repulsive fixed point W_N (the subscript N suggested in Ref. 15 to underline the similarity to the role of the Nishimori point in the RBIM) is located on the critical line separating two insulating phases at the disorder smaller than the one at the tricritical point W_T, where the critical line splits. It has been suggested[16] that we can address this question using our optimization procedure studying the critical exponent for different values of disorder W. Indeed, if one believes the scenario of two fixed points, then the one-parameter scaling for the disorder $W<W_N$ should produce the critical exponents which tend to the value $\nu=1$ of the pure Ising model as one uses only large system widths to get rid of the finite-size

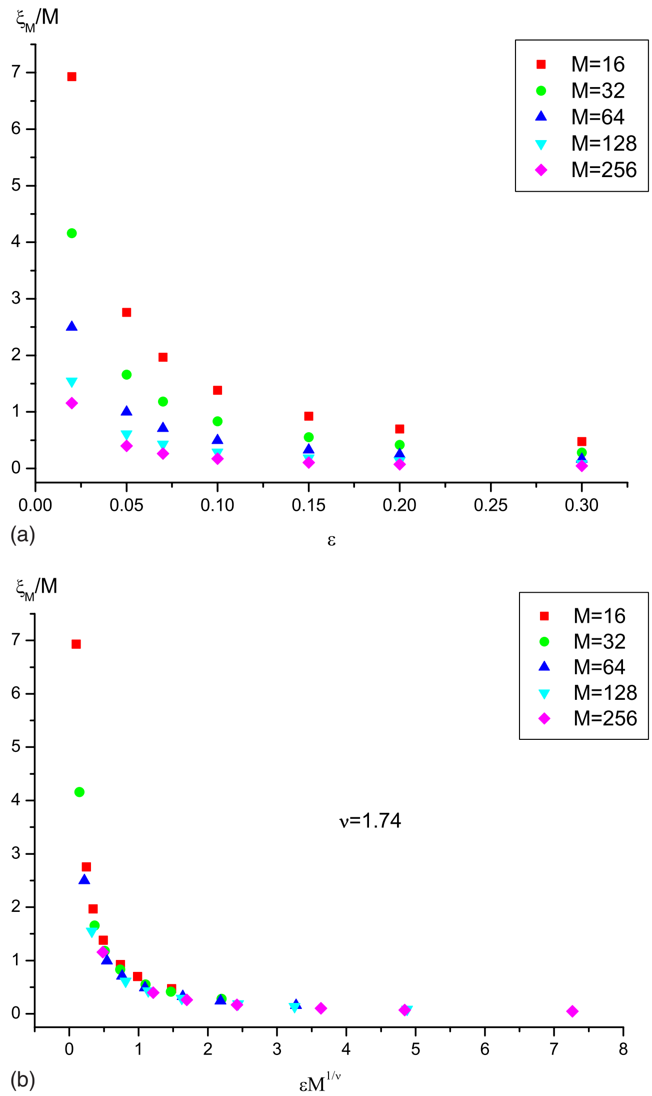

FIG. 3. (Color online) Numerical results for the disorder $W=0.12$. (a) Renormalized localization length ξ_M/M versus energy ϵ for various system widths. (b) A fit of data to one-parameter scaling form $\xi_M/M=f(\epsilon M^{1/\nu})$ for all system widths $M=16,32,64,128,256$ and $\nu=1.74$.

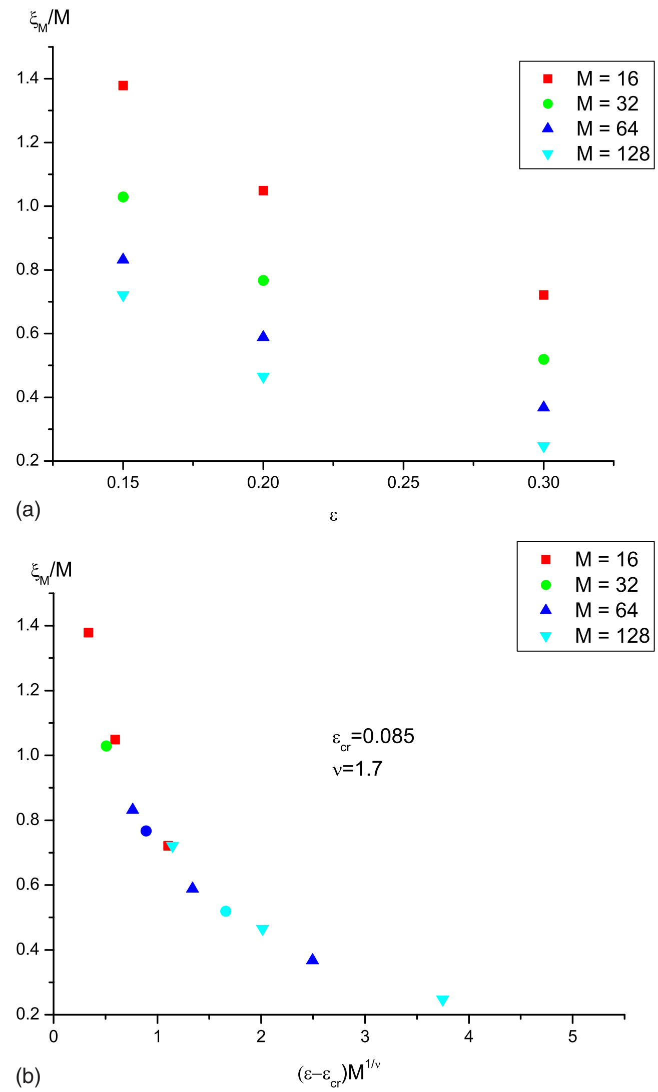

FIG. 4. (Color online) Numerical results for the disorder $W=0.16$. (a) Renormalized localization length ξ_M/M versus energy ϵ for various system widths. (b) A fit of data to one-parameter scaling form $\xi_M/M=f[(\epsilon-\epsilon_{cr})M^{1/\nu}]$ for all system widths $M=16,32,64,128$, $\epsilon_{cr}=0.085$, and $\nu=1.7$.

effects. In contradistinction, for $W>W_N$ the repulsive fixed point should push the critical exponent to the value at the tricritical fixed point W_T.

We have performed numerical calculations at small fixed values of W for the system widths $M=16$ and various energies ϵ. In those cases an obvious critical energy is $\epsilon=0$ (the most quantum case explained above). We have been determining the critical exponents ν ($\xi\sim\epsilon^{-\nu}$) to fit all the data onto one curve according to Eq. (2) by applying a special optimization program which checks different critical exponents and chooses the optimal one. We now briefly describe the optimization procedure.

The routine determines least-squares polynomial approximation by minimizing the sum of squares of the deviations of the data points from the corresponding values of the polynomial. The argument of the function fitted by the Chebyshev polynomials is $M|\epsilon-\epsilon_{cr}|^{\nu}$. We run this routine for the wide range of the ν values and choose the optimal one. When it is necessary, we also look for the optimal value of ϵ_{cr} as discussed below.

We have carried out the analysis, first, for all data points for all system widths, second, without $M=16$ data, and third, without $M=16$ and 32 data. It turns out that, indeed, this procedure shows that the critical exponent approaches the predicted pure Ising model value $\nu=1$ as we omit small system width data. As a typical example we present here the data [Fig. 2(a)] and the values of the critical exponent ν for the disorder $W=0.04$: $\nu=1.34$ for $M=16,32,64,128$ [Fig. 2(b)], $\nu=1.11$ for $M=32,64,128$ [Fig. 2(c)], and $\nu=0.97$ for $M=64,128$ [Fig. 2(d)]. The largest value of disorder for which this tendency persists is 0.1. To demonstrate that in a conclusive way at this disorder value, the data for the largest system width $M=256$ were necessary: the critical exponent changes as $1.7\rightarrow1.6\rightarrow1.2\rightarrow1.1$ when we omit the small system width data consequently. We, therefore, identify $W_N = 0.1$. We argue that our study of the critical exponent can be related to the RG flow. Indeed, when we omit small system width data, the smaller the disorder W is, the quicker the critical exponent converges to the value $\nu = 1$. This qualitatively corresponds to the RG flow from $W_N = 0.1$ to the pure Ising model for $W = 0$. In full agreement with the scenario suggested[15] for $W > W_N$, the critical exponent has a value distinctively different from $\nu = 1$; e.g., for $W = 0.12$ (Fig. 3) the critical exponent is almost constant $\nu \approx 1.7$, independent of the system widths used. In order to be sure that we are still on the critical line, we allow the optimization program to look for the critical energy as well. It is very conclusive that up to the disorder value $W = 0.14$ the critical energy is about $\epsilon_{cr} \approx 10^{-5}$ which supports the single critical line. For $W > 0.14$ the optimization program immediately produces small but finite values of the critical energy. As an illustration we show in Fig. 4 the data for $W = 0.16$ and its one-parameter scaling with $\epsilon_{cr} = 0.84 \neq 0$ and $\nu = 1.7$. We thus identify the tricritical fixed point $W_T = 0.14$. Obviously, nonzero critical energies are not calculated rigorously, which is shown on the phase diagram (Fig. 1) by enlarged "stars."

To summarize, we have studied in detail the critical line on the phase diagram of the CF model. We have calculated the critical exponent and determined two fixed points in agreement with one of the scenarios suggested in Ref. 15: the repulsive fixed point at disorder $W_N = 0.1$ and the tricritical fixed point $W_T = 0.14$. Obviously, we cannot rule out completely that for larger system widths the RG flow can reverse as was found.[15]

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